eRm – Extended Rasch Modeling

An R Package for the Analysis of Extended Rasch Models

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Contents

Rasch Measurement

- **Item Response Theory (IRT)**
  - Analysis of response patterns in tests and questionnaires.
  - Functional relationship between the probability to solve an item and some item and person parameter.
  - A simple model is the Rasch model with one parameter β_i for each item and one parameter θ_v for each person.
  - Rasch model as a seal of approval of a test (fairness, scaling)
- **Rasch measurement scale**
  - Example: A temperature scale in physics is clearly defined. What about a scale for mathematical ability?
  - The Rasch model generates a scale for a latent trait Ψ

The Rasch Model

- **Rasch model equation for dichotomous items**

X

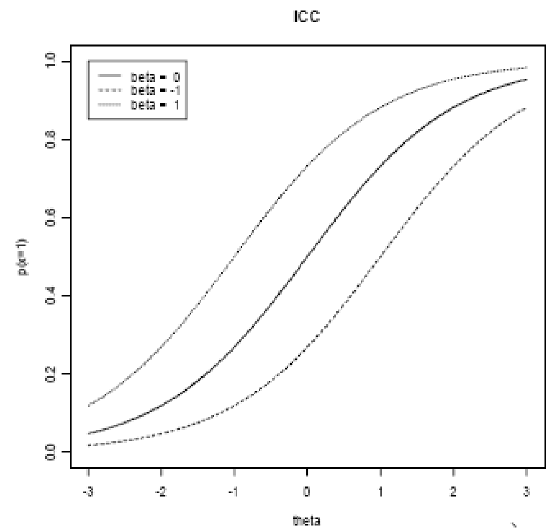
	I_1	I_2	I_3	I_4	I_5	R_v
P_1	1	1	1	1	1	5
P_2	1	1	1	0	1	4
P_3	0	0	1	1	1	3
P_4	1	0	0	0	0	1
P_5	0	0	0	0	0	0
…						

$$P\left(X_{vi}=1 \mid \beta_i, \theta_v\right)=\frac{\exp\left(\theta_v-\beta_i\right)}{1+\exp\left(\theta_v-\beta_i\right)}$$

β_i …item parameter

θ_v …person parameter

- **Assumptions**
  - Unidimensionality
  - Sufficiency of the raw score
  - Parallel item characteristic curves
  - Local independence



Linear Logistic Test Model

- **LLTM (Linear Logistic Test Model)**
  - Linear reparameterization of the Rasch model.

$$\beta_i = \sum_{j=1}^{p} w_{ij}\eta_j$$

  - LLTM as a more parsimonious model.
  - LLTM as a more general model in terms of repeated measurements and certain effects.
  - Concept of virtual items.

$$
\begin{array}{c|cccc:cccc|c}
 & \beta_1 & \beta_2 & \dots & \beta_K & \tau & \delta & \nu & \rho & \\
\beta_1^* & 1 & & & & & & & & \\
\beta_2^* & & 1 & & & & & & & \text{B}_1 \\
\vdots & & & \ddots & & & & & & \\
\beta_K^* & & & & 1 & & & & & \\
\hdashline
\beta_{K+1}^* & 1 & & & & 1 & & & & \\
\beta_{K+2}^* & & 1 & & & 1 & & & & \text{B}_2 \\
\vdots & & & \ddots & & \vdots & & & & \\
\beta_{2K}^* & & & & 1 & 1 & & & & \\
\hdashline
\beta_{2K+1}^* & 1 & & & & 1 & 1 & & & \\
\beta_{2K+2}^* & & 1 & & & 1 & 1 & & & \text{B}_3 \\
\vdots & & & \ddots & & \vdots & \vdots & & & \\
\beta_{3K}^* & & & & 1 & 1 & 1 & & & \\
\hdashline
\beta_{3K+1}^* & 1 & & & & 1 & & 1 & & \\
\beta_{3K+2}^* & & 1 & & & 1 & & 1 & & \text{B}_4 \\
\vdots & & & \ddots & & \vdots & & \vdots & & \\
\beta_{4K}^* & & & & 1 & 1 & & 1 & & \\
\hdashline
\beta_{4K+1}^* & 1 & & & & 1 & 1 & 1 & 1 & \\
\beta_{4K+2}^* & & 1 & & & 1 & 1 & 1 & 1 & \text{B}_5 \\
\vdots & & & \ddots & & \vdots & \vdots & \vdots & \vdots & \\
\beta_{5K}^* & & & & 1 & 1 & 1 & 1 & 1 &
\end{array}
$$

Rating Scale Models

- **RSM (Rating Scale Model)**
  - Item response categories are rating scales (polytomous)

$$P\left(X_{vi}=k \mid \theta_v,\beta_i,\omega_0,\ldots,\omega_m\right)=\frac{\exp\left(k\left(\theta_v+\beta_i\right)+\omega_k\right)}{\sum_{h=0}^{m}\exp\left(h\left(\theta_v+\beta_i\right)+\omega_h\right)}$$

$h,k\ldots$categories; $h=0,\ldots,m$

$\omega_h\ldots$category parameter

- **LRSM (Linear Rating Scale Model)**
  - Linear decomposition of the item parameter

$$\beta_i=\sum_{j=1}^{p} w_{ij}\eta_j$$

Partial Credit Models

- **PCM (Partial Credit Model)**
  - Each item category gets a partial credit (item-category parameter)
  - Different number of categories per item allowed

$$P\left(X_{vik}=1 \mid \theta_v, \beta_{ik}\right)=\frac{\exp\left(k\theta_v+\beta_{ik}\right)}{\sum_{h=0}^{m_i}\exp\left(h\theta_v+\beta_{ih}\right)}$$

β_{ih} …item-category parameter

- **LPCM (Linear Partial Credit Model)**
  - Linear decomposition of the item-category parameter

$$\beta_{ih}=\sum_{j=1}^{p} w_{ihj}\eta_j$$

Model Hierarchy

- **Model Nesting**

Rasch	$\subset$	RSM	$\subset$	PCM		
		$\subset$			$\subset$	LPCM
LLTM	$\subset$	LRSM				

- **LPCM is the most general model**
  - All other models can be viewed as special cases of LPCM.
  - Parameterization through appropriate choice of **W**.
  - Unified CML procedure which is able to estimate these models.

A Unified CML Approach

- **Linear item parameter decomposition**

$$\boldsymbol{\beta} = \mathbf{W}\boldsymbol{\eta}$$

$\mathbf{W}$…design matrix

$\boldsymbol{\eta}$…parameter vector

- **CML approach**
  - Estimation of $\hat{\boldsymbol{\eta}}$
  - ML conditioned on the raw score → $\boldsymbol{\theta}$ vanishes

$$\log L_C = \sum_l \sum_{h_l} x_{+lh_l} \beta_{lh_l} - \sum_r n_r \log \gamma_r(\varepsilon)$$

$$\frac{\partial \log L_C}{\partial \eta_a} = \sum_l \sum_{h_l} w_{lh_l a} \left(x_{+lh_l} - \varepsilon_{lh_l} \sum_r n_r \frac{\gamma_{r-h_l}^{(l)}(\varepsilon)}{\gamma_r(\varepsilon)} \right)$$